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ABSTRACT

This paper presents two extensions of the full-rank multivariate linear model that are particularly useful in multivariate analysis of covariance (MANCOVA) and repeated measurements designs. After a review of the basic full-rank model, an extension is described which allows restrictions of a more general nature. This model is useful in the analysis of repeated measurements designs when the researcher assumes certain interactions do not exist and wishes to pool sources of variation, or in the analysis of repeated measurements designs with covariates. A further extension of the basic model allows an even more general class of restrictions. This extended model allows testing hypothesis on the between groups portion of a repeated measures design with covariates, subject to a restriction of equality of slopes over measures. (Author/BW)

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SOME EXTENSIONS OF THE FULL RANK MULTIVARIATE LINEAR MODEL

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Some Extensions of the Full Rank Multivariate Linear Model

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1. Introduction

Timm and Carlson (1973) presented a full rank formulation of the multivariate linear model that is useful for the analysis of Multivariate Regression, MANOVA, MANCOVA and Discriminant Function Analysis problems. This model is also used by Timm (1975) in his discussion of profile analysis and growth curve analysis. The advantage of this formulation of the model is that it always employs a full-rank design matrix and is therefore mathematically simpler than the less-than-full-rank models employed in most references (Bock, 1975; Timm, 1975, in most of his book), which require such concepts as reparameterization, side conditions, generalized inverses or the determination of orthogonal bases of vector spaces for estimation and hypothesis testing. Additionally the parameters of the model are the parameters of direct interest to the researcher (cell means in MANOVA, slopes and intercepts in MANCOVA) rather than linear functions of these parameters as employed in the less-than-full-rank model.

In this paper we present two extensions of the full-rank model that are particularly useful in MANCOVA and repeated measurements designs. In the next section the basic full-rank model is reviewed. The extensions are then presented in the following sections.

2. The Full Rank Multivariate Linear Model (FRMLM)

The model as presented by Timm and Carlson (1973) is

$$Y = WU + E \tag{2.1}$$

$$R'U = \Theta \quad \text{(optional)}$$

where:

Y = an N by q matrix of observations on N subjects and q variates

W = an N by p nonrandom matrix of full rank, p, containing group membership indicator variables in MANOVA, independent variables in multivariate regression, and both types of variables in MANCOVA

U = a p by q matrix of nonrandom parameters

E = an N by q matrix of errors with rows independently distributed $N_q(0,\Sigma)$

R' = an optional r by p matrix of known constants, having rank r

Θ = an r by q matrix of constants

The restriction equation is optional and is used to specify restrictions on the parameters in U, with Θ ususally, but not necessarily, a matrix of zeros. The restrictions are used to incorporate assumptions such as homogeneity of slopes or no interaction into the model.

Minimizing the least squares criterion

$$(2.2) \qquad \mathrm{Tr}\left[(Y-WU)'(Y-WU)\right]$$

where Tr is the trace operator, the minimum variance linear unbiased estimator of U in the unrestricted model is

$$(2.3) \qquad \hat{U} = (W'W)^{-1}W'Y$$

Replacing U by U_r for restricted models, the estimator subject to the restriction is

$$(2.4) \qquad \hat{U}_r = \hat{U} - (W'W)^{-1}R\left[R'(W'W)^{-1}R\right]^{-1}(R'\hat{U} - \Theta)$$

To obtain an unbiased estimator of Σ under the unrestricted model the matrix of sums of squares and products (SSP) about the model

$$(2.5) \qquad \hat{Q} = (Y - W\hat{U})'(Y - W\hat{U})$$

is divided by its degrees of freedom, $\nu_e = N-p$,

$$(2.6) \qquad S = \hat{Q}/\nu_e$$

In the restricted model, similarly,

$$(2.7) \qquad \hat{Q}_r = (Y - W\hat{U}_r)'(Y - W\hat{U}_r)$$

$$= \hat{Q} + (R'\hat{U} - \Theta)'\left[R'(W'W)^{-1}R\right]^{-1}(R'\hat{U} - \Theta)$$

which is divided by its degrees of freedom, $\nu_{er} = N-p+r$, to get

$$(2.8) \qquad S_r = \hat{Q}_r / \nu_{er}$$

the estimator of Σ under the restricted model.

Hypotheses of the form

$$(2.9) \qquad H: C'UA = \Gamma$$

where:

C' = a ν_h by p matrix of known constants, of rank ν_h. The rows of C' must be linearly independent of the rows of R' in restricted models.

A = a q by u matrix of known constants, of rank u

Γ = a ν_h by u matrix of known constants

The pre-matrix, C' is used to specify contrasts between the rows of U while A is used to specify contrasts between the columns. In certain hypotheses either C' or A is the identity matrix. Γ is usually, but not necessarily a matrix of zeros.

For unrestricted models the SSP matrix for the hypothesis is

$$(2.10) \qquad S_h = (C'\hat{U}A - \Gamma)'\left[C'(W'W)^{-1}C\right]^{-1}(C'\hat{U}A - \Gamma)$$

while the error SSP matrix is

$$(2.11) \qquad S_e = A'\hat{Q}A$$

When the assumptions on E in (2.1) are met, and the null hypothesis is true, S_h and S_e are independently distributed as central Wishart random matrices:

(2.12) $S_h \sim W_u(\nu_h, A'\Sigma A)$

$S_e \sim W_u(\nu_e, A'\Sigma A)$

Letting $\lambda_1, \lambda_2, \ldots, \lambda_s$ and $v_1, v_2, \ldots, v_s$, be the roots and vectors of the determinantal equation,

(2.13) $|S_h - \lambda S_e| = 0$

one of the following multivariate test criteria is used to test H (see, for example, Timm, 1975):

(2.14)

Wilks' Lambda	$\Lambda = \prod_{i=1}^{s} (1 + \lambda_i)^{-1}$
Roy's Largest Root	$\theta = \lambda_1/(1 + \lambda_1)$
Lawley-Hotelling Trace	$U^{(s)} = \sum_{i=1}^{s} \lambda_i$
Pillai's Trace	$V^{(s)} = \sum_{i=1}^{s} \lambda_i/(1 + \lambda_i)$

Tables and charts of critical values of these statistics may be found in Timm (1975).

When the null hypothesis is rejected Porebski (1966) and Bargmann (1970) suggest computing correlations between the original variates in Y (or transformed variates YA when A in the hypothesis is not an identity matrix) and the generalized discriminant functions (canonical variates). Letting the vector

(2.15) $b_i = v_i/\sqrt{v_i' S_e v_i} \qquad (i = 1, 2, \ldots, s)$

denote the coefficients of the standardized canonical variates (variance one in the sample), these variates are

(2.16) $L_i = Yb_i \qquad (i = 1, 2, \ldots, s)$

with Y replaced by YA when A is not the identity matrix. Letting R_e be the error correlation matrix,

(2.17) $$R_e = (\text{Dia } S_e)^{-\frac{1}{2}} S_e (\text{Dia } S_e)^{-\frac{1}{2}}$$

and d be the vector of diagonal elements of S_e, the correlations between the original (Y) or transformed (YA) variates and the canonical variates are the vectors

(2.18) $$r_i = R_e f_i \qquad (i = 1, 2, \ldots, s)$$

where f_i is the Hadamard vector product

$$f_i = b_i * d$$

The canonical variates, L_i, represent those linear functions of the variates in Y or YA which are most sensitive to deviations from the null hypothesis. Plotting the group centroids on the canonical variates in the discriminant space (Timm, 1975) also aids interpretation.

Bartlett's asymptotic chi squared statistic is often suggested to test the significance of the successive discriminants but a test due to Glynn (cited by Chou and Muirhead, 1979) is superior. This statistic is

(2.19) $$L_{s'} = -\left[(\nu_e - s' + (\nu_h - u - 1)/2 + \sum_{i=1}^{s'} \lambda^{-1}\right] \ln \left[\prod_{i=s'+1}^{s} (1-\lambda)\right]$$

which tests the hypothesis that roots s'+1 through s equal zero given roots 1 through s. This statistic is asymptotically distributed as chi squared with degrees of freedom $(\nu_h - s')(u - s')$.

As an alternative to the generalized discriminant function approach, confidence intervals for parametric functions of the elements in U may be examined to aid in the interpretation of a significant multivariate test statistic. The confidence interval is

(2.20) $$\hat{\psi} - c_o \hat{\sigma}_\psi \leq \psi \leq \hat{\psi} + c_o \hat{\sigma}_\psi$$

where:

(2.21) $\hat{\psi} = c'Ua$

$\hat{\sigma}^2_{\hat{\psi}} = (a'Sa)\, c'(W'W)^{-1}c$

and c_o is defined for each criterion by the following relationships (Gabriel, 1968, 1969)

(2.22)

Wilks	$c_o^2 = \nu_e(1-U^\alpha)/U^\alpha$
Roy	$c_o^2 = \nu_e\theta^{(1-\alpha)}/\left(1-\theta^{(1-\alpha)}\right)$
Lawley-Hotelling	$c_o^2 = \nu_e U_o^{(1-\alpha)}$
Pillai	$c_o^2 = \nu_e V^{(1-\alpha)}/\left(1-V^{(1-\alpha)}\right)$

Parametric functions of interest to the researcher are chosen by choosing appropriate c' and a vectors such as those used in forming C' and A in the hypothesis. Another approach is to use the Bonferroni t statistic. If k parametric functions are to be examined by this approach

(2.23) $c_o = t^{\alpha/2k}_{\nu_e}$

and tabled values may be found in Bailey (1977).

For restricted models the SSP matrix for the hypothesis is shown by Timm and Carlson (1973) to be

(2.24) $S_{hr} = (C'\hat{U}_r A - \Gamma)'(C'FDF'C)^{-1}(C'\hat{U}_r A - \Gamma)$

where: $D = (W'W)^{-1}$

$F = I - DR(R'DR)^{-1}R'$

while the error SSP matrix associated with the hypothesis is

(2.25) $S_{er} = A'\hat{Q}_r A$

with $\hat{Q}_r$ defined in (2.7). All of the procedures outlined above for

the unrestricted model may also be used for the restricted model. In all expressions S_e is replaced by S_{er} and ν_e by $\nu_{er} = N-p+r$. Glynn's statistic in (2.19) is, for example modified by changing ν_e to ν_{er}. In the restricted model the R' matrix must be of full row rank, r, and all rows of C' must be linearly independent of all rows of R' in order that the inverse in (2.24) exists. This requirement simply means that the hypothesis cannot be either identical to or inconsistent with the restrictions.

3. The FRMLM with a more General Restriction

Timm (1979) presented a model that is essentially identical to (2.1) except that restrictions of a more general form are allowed. The model is

$$\begin{aligned} Y &= WU + E \\ R'UA &= \Theta \end{aligned} \tag{3.1}$$

He discussed testing hypotheses of the form (2.9), repeated here for convenience

$$H: C'UA = \Gamma \tag{3.2}$$

under (3.1). Note that the post-matrix, A, in the restriction equation is identical to that in the hypothesis in this model. Under this model the restricted estimator of U is (with $D=(W'W)^{-1}$)

$$\hat{U}_{r2} = \hat{U} - DR(R'DR)^{-1}(R'\hat{U}A - \Theta)(A'A)^{-1}A' \tag{3.3}$$

and A must be of full column rank. The SSP matrix about the model is

$$\hat{Q}_{r2} = \hat{Q} + A(A'A)^{-1}(R'\hat{U}A - \Theta)'(R'DR)^{-1}(R'\hat{U}A - \Theta)(A'A)^{-1}A' \tag{3.4}$$

The hypothesis SSP matrix for this model is identical to that in (2.24) and the error SSP matrix is (2.25) with $\hat{Q}_r$ replaced by $\hat{Q}_{r2}$. The error degrees of freedom are again $\nu_{er} = N-p+r$ and all of the procedures

discussed in Section 2 may be used with this model.

This model is useful in the analysis of repeated measurements designs when the researcher assumes certain interactions do not exist and wishes to pool sources of variation. In the design with three independent groups and four measures, for example, the unrestricted parameter matrix is a 3 by 4 matrix of means for the three groups on the four measures. Then choosing, for example

(3.5) $$R' = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in (3.1) the restriction of no groups by measures interaction is imposed on the model. Under this restriction the sums of squares and products associated with the interaction are assumed random variation and (3.4) pools this variation with $\hat{Q}$. The estimates of the parameters of the model are given by (3.3) and these estimates display zero interaction, consistent with the restricted model. The multivariate hypothesis of no differences among the measures mean vectors under the assumption of no interaction is then tested by choosing, for example

(3.6) $$C' = I_3, \quad A \text{ as in (3.5)}, \quad \Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in (3.2). It should be noted that the multivariate hypothesis of no differences among the group mean vectors cannot be tested from this no-interaction model because that hypothesis uses an identity matrix as the post-matrix, A. If the researcher wishes to use a univariate F-test rather that the multivariate test for the test

on measures the A matrix is orthonormalized and the univariate sums of squares computed as the traces of the hypothesis and error SSP matrices (Bock, 1975; also discussed by Carlson, 1980).

Another example of the usefulness of this restricted model is in the analysis of repeated measures designs with covariates. If a single covariate measure is added to the 3-group by 4-measures design discussed above the parameter matrix for the unrestricted model is

(3.7)
$$U = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix}$$

where α_{ij} is the intercept parameter for the regression of the j^{th} measure onto the covariate in the i^{th} group, and β_j is the slope parameter, assumed homogeneous over groups. If we wish to assume equality of the slope parameters over measures we may choose, for example

(3.8)
$$R' = (0 \quad 0 \quad 0 \quad 1), \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad \Theta = (0 \quad 0 \quad 0)$$

We may then test the multivariate hypothesis of no measures effect by choosing, for example

(3.9)
$$C' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ A as in (3.8)}, \quad \Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Similarly, the multivariate test of no measures by groups interaction may be performed by choosing, for example

$$(3.10) \qquad C' = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \text{ A as in (3.8), } \Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Much more detail on this analysis and recovering the univariate statistics *from the* multivariate model is presented by Carlson (1980). The writers are not aware of any other currently used multivariate procedure that allows incorporation of the assumption of equality of the slope parameters in (3.7).

4. A Further Generalization

A model that allows a more general class of restrictions than that presented in section 3 is

$$(4.1) \qquad \begin{aligned} Y &= WU + E \\ R'UT &= \Theta \end{aligned}$$

where T is a q by t matrix of known constants. The hypothesis is again

$$(4.2) \qquad H: C'UA = \Gamma$$

Unlike the model in Section 3 this model does not require that the post-matrix in the restriction be identical to that in the hypothesis.

The derivations associated with this model are outlined below. Let

$$(4.3) \qquad F = \operatorname{tr}\left[(Y-WU)'(Y-WU)\right] + 2\operatorname{tr}\left[\Delta'(R'UT-\Theta)\right]$$

where Δ' is a matrix of Lagrange multipliers. Then the partial derivatives of F with respect to U and Δ are

$$(4.4) \qquad \begin{aligned} \partial F/\partial U &= -2(W'Y) + 2(W'W)U + 2R\Delta T' \\ \partial F/\partial \Delta &= 2(R'UT-\Theta) \end{aligned}$$

setting the partial derivatives to zero yields the equations for the estimator of U:

$$(W'W)\hat{U}_{r3} + R\Delta T' = W'Y$$

(4.5)

$$R'\hat{U}_{r3}T = \Theta$$

Letting $D = (W'W)^{-1}$ the first equation yields

(4.6) $\quad \hat{U}_{r3} = DW'Y - DR\Delta T' = \hat{U} - DR\Delta T'$

Substituting (4.6) into the second equation of (4.5) yields

(4.7) $\quad R'\hat{U}T - R'DR\Delta T'T = \Theta$

which we solve for Δ,

(4.8) $\quad \Delta = (R'DR)^{-1}(R'\hat{U}T - \Theta)(T'T)^{-1}$

Substitution of (4.8) into (4.6) yields the restricted estimator of U as

(4.9) $\quad \hat{U}_{r3} = \hat{U} - DR(R'DR)^{-1}(R'\hat{U}T - \Theta)(T'T)^{-1}T'$

The SSP matrix about the model is then

(4.10) $\quad \hat{Q}_{r3} = (Y - W\hat{U}_{r3})'(Y - W\hat{U}_{r3})$

$$= (Y - W\hat{U})'(Y - W\hat{U}) + (\hat{U} - \hat{U}_{r3})'W'W(\hat{U} - \hat{U}_{r3})$$

$$= \hat{Q} + T(T'T)^{-1}(R'\hat{U}T - \Theta)'(R'DR)^{-1}(R'\hat{U}T - \Theta)(T'T)^{-1}T'$$

Note that T must be of full column rank, t. The expectation is

(4.11) $\quad E(\hat{Q}_{r3}) = (N-p)\Sigma + T(T'T)^{-1}E\left[(R'\hat{U}T-\Theta)'(R'DR)^{-1}(R'\hat{U}T-\Theta)\right](T'T)^{-1}T'$

Since T is of full rank the expectation in the last term of (4.11) is a central Wishart random variable, that is, distributed as $W_t(r, \Gamma'\Sigma T)$.

From (4.10) we see that this term is

(4.12) $\quad T'(\hat{Q}_{r3} - \hat{Q})T$

Thus

(4.13) $\quad T'\hat{Q}_{r3}T = T'\hat{Q}T + W^*$

where W^* is the central Wishart random variable. This implies that

(4.14) $\hat{Q}_{r3} \sim W_t(N-p+r, \Sigma)$

This gives us the mechanism for testing the hypothesis in (4.2) under the model in (4.1). Let P' and B be the partitioned matrices

(4.15) $P' = \begin{bmatrix} R' \\ \hline C' \end{bmatrix}$ $\qquad B = (\, T \,|\, A \,)$

and let $R'UA = \Psi$ and $C'UT = \Pi$. Then we need to minimize

(4.16) $\mathrm{Tr}\left[(Y - WU)'(Y - WU)\right]$

subject to

(4.17) $P'UB = \Psi$

where

(4.18) $\Psi = \left[\begin{array}{c|c} \Theta & \Psi \\ \hline \Pi & \Gamma \end{array}\right]$

The minimization leads to the following expressions for the error and hypothesis SSP matrices:

(4.19) $S_{er3} = \hat{Q}_{r3}$

(4.20) $S_{hr3} = B(B'B)^{-1}(P'\hat{U}B-\Psi)'(P'DP)^{-1}(P'\hat{U}B-\Psi)(B'B)^{-1}B'$

$\qquad - T(T'T)^{-1}(R'\hat{U}T-\Theta)'(R'DR)^{-1}(R'\hat{U}T-\Theta)(T'T)^{-1}T'$

The test statistics discussed in Section 2 can then be computed from the determinantal equation

(4.21) $\left|S_{hr3} - \lambda S_{er3}\right| = 0$

The main purpose of the derivation of this model was to be able to test hypotheses in which the post matrix, A, differed from the post matrix, T, in the restrictions. This would allow, for example, testing hypotheses on the between groups portion of a repeated measures design with covariates, subject to a restriction of equality of

slopes over measures. The motivation for desiring such a model is that the between groups test involves a model based on a linear combination of the measures (Zalar & Carlson, 1980; Carlson, 1980), and the problem of interpretation of such a combination when the slopes differ.

There is a major problem in attempting to apply this model as may be seen be examining (4.17) and (4.18). The Ψ matrix contains known constants, which implies that one knows the values of K and Π in the expressions

$$(4.22) \qquad \begin{aligned} R'UA &= K \\ C'UT &= \Pi \end{aligned}$$

and we currently see no obvious way of specifying these values. We intend to explore these models further to see if there is a way of resolving this problem.

5. Some Similar Models.

Timm (1975) reviewed generalizations of the MANOVA model by Potthoff and Roy (1964), Rao (1965, 1966, 1967) and Khatri (1966), useful in growth curve analysis. These models are of the form

$$(5.1) \qquad Y = XBP + E$$

where BP represents polynomial regressions of Y onto time points for each group defined in the design matrix, X. Hypothesis tests of the form

$$(5.2) \qquad H: C'BA = \Gamma$$

are also reviewed by Timm. These tests involve some reduction of (5.1), either to the form of a MANOVA model,

$$(5.3) \qquad Y = XB + E$$

(Potthoff-Roy), or to the form of a MANCOVA model,

(5.4) $Y = XB + Z\Gamma + E$

(Rao-Khatri). Gleser and Olkin (1970) presented a likelihood ratio criterion for testing (5.2) under the model (5.1). Kabe (1975) generalized the Gleser-Olkin procedure to models containing restrictions of the form discussed in Section 4. In the notation of this paper we write Kabe's model as

(5.5) $Y = XBP + E$

$R'BT = \textcircled{H}$

and hence Kabe's model is more general than ours since it contains the P matrix. Tests of hypotheses of the form (5.2) are discussed by Kabe who employs a statistic that is a product of F-type statistics and is not currently tabled, to our knowledge. Also Kabe's model is subject to the same problem as the model we present in Section 4 -- it is necessary to know certain constants and there is no clear way of specifying the values of these constants.

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